

# Experimentally Determining Passivity Indices: Theory and Simulation

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## Abstract

The passivity index framework is an alternative method of characterizing energy dissipation in systems. As an analysis tool, it can be used to assess the level of passivity of a system. This opens up a much larger class of systems that can be analyzed using results that are similar to the passivity theorem. Typically indices are considered analytically for systems with an established model. This paper focuses on an experimental method of determining indices from input-output data. Particularly, we consider testing passivity indices for vehicle systems with the adaptive cruise control (ACC) algorithm, and maximizing passivity indices through a numerical optimization method, the Hooke and Jeeve's method. Simulations on a virtual car platform are given to demonstrate the results.

# 1 Introduction

Passivity is a dynamic system characterization based on energy dissipation. A passive system is one that does not generate energy, but only stores and dissipates energy provided by the environment. The notion of energy is allowed to be general as in, it is not constrained to any physical notion of energy. This generalized energy is captured by an energy storage function. Passivity is a special case of dissipativity. The benefit of passivity is that when two passive systems are interconnected in parallel or in feedback, the overall system is still passive. Thus passivity is preserved when large-scale systems are created from components that are passive. However this theory is only applicable to systems that are passive. Passivity indices represent an alternative approach to passivity. They can be used to measure an excess or shortage of passivity in a particular system by assessing the feedback and feedforward gains required to render the system passive [1]. As an analysis tool, there are some existing result to assess stability of a single system as well as of feedback interconnections using the passivity index framework.

Methods of determining the passivity indices of a system have been reported in the literature. For instance, it is possible to formulate the search as a traditional LMI optimization problem for linear systems [2]. For general nonlinear systems, passivity indices may exist for a system, but it may be difficult to find their values analytically. The main contribution of this paper is methods of determining passivity indices with experimental testing. Initially, any indices that meet the conditions are considered. This paper then moves on to discuss numerical optimization methods to maximize the passivity indices. Since this approach does not require a model of a system, traditional optimization methods using a gradient are not applicable. Instead, Hooke and Jeeve's method [13] is used as it is a derivative-free numerical optimization approach. Simulations of this method for a virtual car platform are provided to validate the results.

The paper is organized as follows. In Section II, background on passivity and passivity indices is covered, as well as traditional methods of determining passivity. In Section III, an experimental test for passivity is given, followed by an experimental passivity test for vehicle systems with adaptive cruise control (ACC) in Section IV. In Section V, numerical methods for experimental passivity optimization is discussed with simulation results demonstrated. Finally, concluding remarks are made in Section VI.

## 2 Background on Passivity and Passivity Indices

### 2.1 Defining Passivity and Passivity Indices

This paper will use notions of passivity and passivity indices extensively. While these notions can be defined for state based systems, this paper will focus mostly on the input-output definition of these concepts. Before these definitions are introduced, the signals and systems of interest will be defined.

An  $m$ -dimensional continuous time signal  $u(t)$  is a mapping from the positive time axis  $\mathbb{R}^+$  to the space  $\mathbb{R}^m$ . If this signal has finite energy over all time, it is considered to be in an  $L_2$  signal. It has  $L_2$ -norm given by the expression,

$$\|u\|_2^2 = \int_0^\infty u^T(t)u(t)dt < \infty. \tag{1}$$

While the space of signals with finite energy is useful, it is not possible to consider unstable systems without considering a more general space. The extended signal space,  $L_{2e}$ , is the set of signals with finite energy on any finite time interval. A continuous-time signal  $u : \mathbb{R}^+ \rightarrow \mathbb{R}^m$  is in  $L_{2e}$  if

$$\|u_T\|_2^2 = \int_0^T u^\top(t)u(t)dt < \infty, \quad \forall T \in \mathbb{R}^+. \quad (2)$$

A system  $H$  is a mapping from input  $u \in U \subset L_{2e}$  to output  $y \in Y \subset L_{2e}$ , where  $y \in \mathbb{R}^m$ . If  $u$  is a given element of  $L_{2e}$ , then  $Hu$  denotes an image of  $u$  under  $H$ , where  $y = Hu$ . The notion of stability used in this paper is  $L_2$  stability.

**Definition 1** *A system  $H : L_{2e} \rightarrow L_{2e}$  is  $L_2$  stable if*

$$u \in L_2 \implies y \in L_2.$$

An important class of  $L_2$  stable systems is the class of systems with finite  $L_2$ -gain. This concept can be captured by the following input-output condition.

For all time  $T \in \mathbb{R}^+$  and for all inputs  $u \in L_{2e}$ , a system  $H$  is finite-gain  $L_2$  stable if there exist constants  $\gamma > 0$  and  $\beta$  such that

$$\|y_T\|_2 \leq \gamma \|u_T\|_2 + \beta. \quad (3)$$

With the appropriate background material presented, the definitions of passivity and passivity indices can be presented. A system is passive if it only stores and dissipates energy without generating its own energy. This is captured by an inequality where the energy supplied to the system by its environment from initial time to time  $T$ ,  $\int_0^T y^\top(t)u(t)dt$ , is an upper bound on the loss of initially stored energy,  $-\beta$ .

**Definition 2** *For a system  $H$ , consider all inputs  $u \in U \subset L_{2e}$  and all times  $T \in \mathbb{R}^+$ .  $H$  is passive if  $\exists \beta \geq 0$  such that*

$$\int_0^T y^\top(t)u(t)dt \geq -\beta. \quad (4)$$

An alternative approach to energy-based analysis of dynamical systems is the passivity index framework [1]. While passivity is only a binary property, a system is passive or not, the passivity indices capture the level of passivity present in a dynamical system. For example, a system that is not passive may be “nearly” passive in the sense that a small feedback or feed-forward gain will make the system passive. Likewise it can be useful to distinguish between systems that just passive and ones that are “excessively” passive, i.e. dissipate more energy than necessary. The benefit of using these indices is that a system that is by some measure “nearly” passive may be compensated by a system in feedback that is “excessively” passive.

The concept of indices came from applying earlier work of conic systems [3, 4] to state space systems. A detailed survey of passivity indices can be found in [1]. Passivity indices can be defined for general nonlinear systems in the same framework as passivity.

**Definition 3** [1, 5] *For a system  $H$ , consider all inputs  $u \in U \subset L_{2e}$  and all times  $T \in \mathbb{R}^+$ .  $H$  has output feedback passivity (OFP) index  $\rho$  and input feed-forward passivity (IFP) index  $\nu$  if there exists a constant  $\beta$  such that the following inequality holds,*

$$\int_0^T \left[ (1 + \rho\nu)u^\top y - \rho y^\top y - \nu u^\top u \right] dt \geq -\beta, \quad (5)$$

## 2.2 Finding Indices using LMIs

It will be important in this paper to compare the results of experimental passivity to traditional methods of determining passivity or finding indices from models. This can be done for LTI systems using linear matrix inequalities (LMIs). Continuous time LTI systems can be defined by the state space model:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (6)$$

$$y(t) = Cx(t) + Du(t). \quad (7)$$

It will be assumed that this model is minimal, i.e. controllable and observable.

Passivity can be shown for state based systems by defining an energy storage function  $V(x(t))$ . This function is a measure of the internally stored energy of a system. As a generalized notion of energy, the storage function is non-negative everywhere,  $V(x(t)) \geq 0, \forall x$ . It will be assumed to be zero at the equilibrium. Without loss of generality, this point can be assumed to be  $x = 0$ . Also without loss of generality, for LTI systems it can be assumed that  $V$  has a quadratic form,

$$V(x(t)) = x(t)^T Px(t), \quad (8)$$

where  $P = P^T$ . The condition that  $V(x(t))$  be positive can be reduced to the matrix  $P$  being positive semi-definite,  $P \geq 0$ .

For LTI systems, a necessary and sufficient test for passivity indices to hold is that  $\exists P = P^T > 0$  such that the following LMI is satisfied:

$$\begin{bmatrix} A^T P + PA + \rho C^T C & PB - \frac{1}{2}(1 + \rho\nu)C^T + \rho C^T D \\ (PB - \frac{1}{2}(1 + \rho\nu)C^T + \rho C^T D)^T & \rho D^T D - (1 + \rho\nu)(D + D^T) + \nu I \end{bmatrix} \leq 0. \quad (9)$$

The LMI can be used for passivity by setting both  $\rho = 0$  and  $\nu = 0$ . Assuming  $\rho$  and  $\nu$  are fixed, the LMI is linear in the decision variable ( $P$ ) so can be solved using traditional LMI optimization methods [2].

## 3 Experimental Passivity

The traditional method of control system design involves modeling the plant to be controlled, analyzing the plant, and then synthesizing a controller. Using passivity theory or theory from passivity indices does not require the use of a model. Instead, data can be collected to determine that a system is passive or that it has certain passivity indices. In most cases, this data collection must be significantly thorough in order to make these conclusions. However, the amount of data required is similar to the amount of data required to determine and verify a model.

As in Definition 2, a system is considered passive if it satisfies the following inequality for all inputs  $u(t)$  in a set  $U$ ,

$$\int_0^T y^T(t)u(t)dt \geq -\beta. \quad (10)$$

To satisfy this definition, the system must satisfy the inequality for all inputs  $u$  as well as all finite times  $T$ . The condition should hold for  $\beta$  that can depend on the initial condition and must not depend on the time  $T$ . Of course, it is impossible to test for arbitrarily large  $T$ . For example,

for non-passive unstable systems there always exists a finite  $\beta$  to satisfy the inequality for  $T$  up to a given time. However, as  $T$  goes to infinity, the bound involving a finite  $\beta$  will not hold for non-passive systems.

Alternatively, the inequality can be changed to test for passivity under zero initial conditions. In this case, the constant  $\beta$  can be taken to be zero. The inequality is then,

$$\int_0^T y^\top(t)u(t)dt \geq 0. \quad (11)$$

In this case, if the inner product of  $u$  and  $y$  is negative for any  $T$ , the test fails. For systems that are passive with respect to the particular input, the inner product will vary with the input but on average will grow without bound. If this pattern holds for a sufficiently long initial time interval, it can be concluded that the inner product will not suddenly differ from this trend unless the input suddenly changes with regard to signal magnitude, frequency, etc. Depending on the input, this initial time interval may be shorter or longer.

There are a couple limitations to note with this approach. One is that this experimental result is a sufficient only test for passivity in a finite duration of time. Another is that the input set  $U$  is often not a finite set in practice. An actual input to a system is likely not to be exactly specified in advance so will not be contained within a finite set. When the set  $U$  contains an infinite set of inputs, the test for passivity can be modified. A finite subset can be chosen that represents the diversity of the set  $U$  in terms of signal magnitude, frequency content, etc.

A similar approach can be taken to estimate passivity indices for a system. In this case the inequality to be satisfied takes the following form,

$$(1 + \rho\nu) \int_0^T y^\top(t)u(t)dt \geq \rho \int_0^T y^\top(t)y(t)dt + \nu \int_0^T u^\top(t)u(t)dt. \quad (12)$$

When testing this inequality, the indices can be estimated from each data set and for each time  $T$ . When considering all data, this gives many constraints on the indices. Any algorithm that provides a final set of indices from this data must give indices that are less than all the bounds. How the exact indices are chosen from the bounds is another problem. It is often possible to reduce one index in order to increase the other.

It should be noted that the indices are necessarily going to be larger for a restricted set of inputs than for any possible input in  $L_{2e}$ . Additionally, the algorithm may give bounds that are not tight for the indices. In this case, a buffer should be introduced to give a more conservative bound on the indices. This is especially important in the practical case when a representative sample of inputs is tested rather than the full set. In the following section, the experimental passivity indices will be tested for the vehicle performance of the adaptive cruise control(ACC) algorithm.

## 4 Experimental Passivity for Vehicles with ACC

### 4.1 Modeling Environment

The vehicle modeling environment used to facilitate simulations and to present the passivity-based optimization is based on MATLAB [6] and CarSim [7].

MATLAB is a programming environment produced by MathWorks for algorithm development, data analysis, visualization, and numerical computation. Simulink[3] is an environment for multidomain simulation and Model-Based Design for dynamic and embedded systems. It provides an

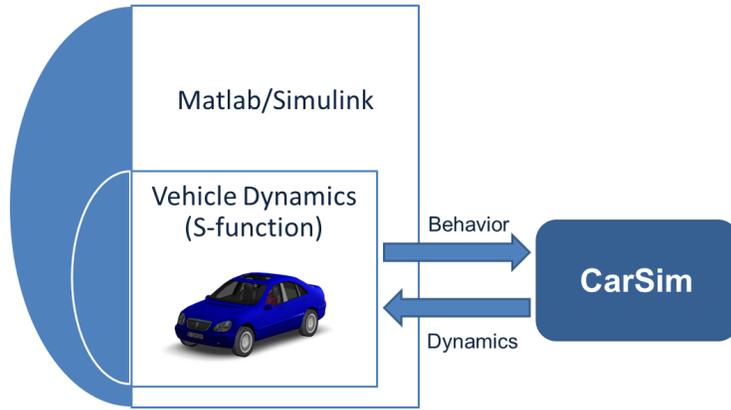


Figure 1: Simulation environment

interactive graphical environment and a customizable set of block libraries that let you design, simulate, implement, and test a variety of time-varying systems, including communications, controls, signal processing, video processing, and image processing.

CarSim is a software package that simulate the dynamic behavior of vehicles. In response to driver controls such as throttle, brakes and steering, the performance of vehicles can be analyzed in various road environment. CarSim animates simulated tests and outputs over hundreds of calculated variables to plot and analyze, or export to other software such as MATLAB, Excel, and optimization tools.

An s-function application programming interface (API) establishes the interface between CarSim and MATLAB/Simulink. All parameters and dynamics of the vehicle is imported into Simulink model, before the Simulink model typically involves underlying differential equations that are solved by Simulink using numerical methods. After simulation is finished in Simulink, data is sent back to CarSim so that a visualized animation can be displayed with CarSim animation viewer SurfAnim. Fig. 1 shows the overview of our modeling environment.

## 4.2 Model for ACC

The adaptive cruise control(ACC) algorithm is based on [8], where two hierarchical levels of control is applied. The upper level controller computes the desired acceleration for the ACC-equipped vehicle that achieves the desired spacing or velocity. The lower level controller determines whether to apply braking control or throttle control through a switching logic component, and then a brake torque or engine torque is computed to achieve the desired acceleration. More details can be found in [8].

## 4.3 Simulation Results

In the following, the passivity index of the host vehicle system is estimated through simulations. Fig. 2 shows a scenario that the host vehicle is collecting information of the lead vehicle and is following. For the host vehicle system, the velocity of the lead vehicle can be seen as the system input  $u$ , which the velocity of the host vehicle can be seen as the system output  $y$ . Both input and

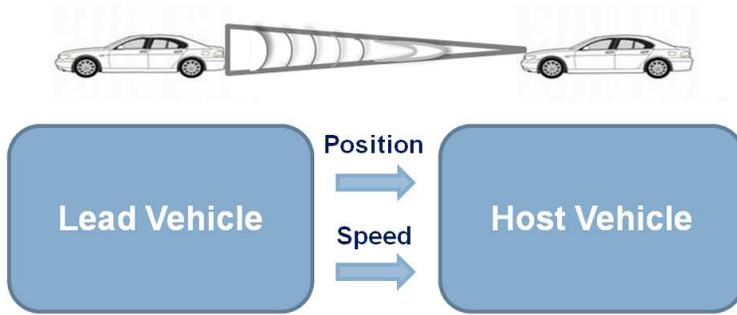


Figure 2: Two Vehicle Scenario

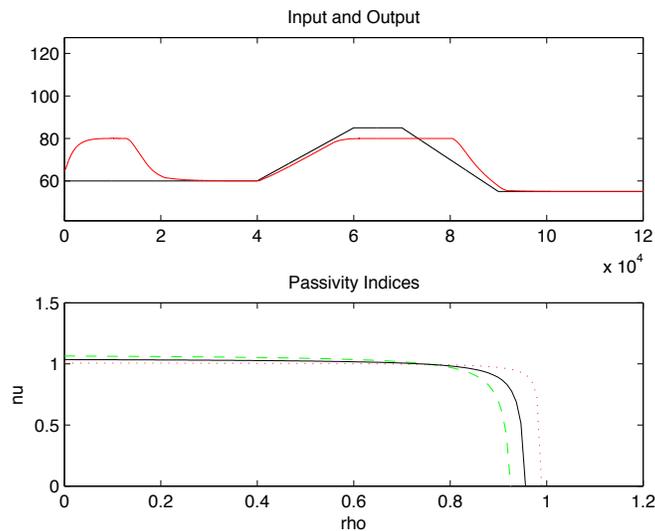


Figure 3: Velocities of Two Cars and Passivity Indices

output are discrete-time vectors with the same length, therefore the experimental passivity test (12) in Section III can be applied.

In our case, as  $u$  and  $y$  are always non-negative for any  $T$ , the system will always be passive. However the inner product of  $u$  and  $y$  will vary with respect to input, i.e. behavior of the lead vehicle. With different input data set and different selected time intervals, the estimation of the passivity indices  $\rho$  and  $\nu$  may vary. Therefore this estimation has its limitation if a sufficiently long time interval is not given. But the input set  $U$  or the lead vehicle velocity set in this case is bounded by mechanical constraints. We can choose a finite subset of the input set to represent the diversity of the set  $U$  in terms of signal magnitude, frequency content, etc., and estimate the indices accordingly. The estimation may give different bounds for the indices with different time interval and different samples of inputs, but with sufficient large data sample the bound on the indices should be relatively conservative.

Fig. 3 show the simulation results of the example in [8]. In the top figure, velocities of the two cars are plotted. The lead car has the initial velocity of 60 km/h, and then it accelerates during

	0s to 60s	60s to 120s	0s to 120s
$\rho$	0.9254	0.9895	0.9563
$\nu$	1.0643	1.0069	1.0349

Table 1: Experimental Passivity with Sample Data

the time of 40s to 60s, and decelerates during 70s to 90s. The host car is catching up with the lead car before 20s, and then it tries to follow the lead vehicle.

In the bottom figure, three estimated boundaries of  $\rho$  and  $\nu$  pair are plotted. For the dashed line, only input and output data from 0s to 60s is considered. For the red dotted line, only data from 60s to 120s is considered. And for the solid line, data from 0s to 120s is considered. In the curves, every point representing a  $(\rho, \nu)$  pair makes equality holds in (12), i.e.

$$(1 + \rho\nu) \int_0^T y^\top(t)u(t)dt = \rho \int_0^T y^\top(t)y(t)dt + \nu \int_0^T u^\top(t)u(t)dt. \quad (13)$$

At the intersection of the curves and axes, the output feedback passivity(OFP) index  $\rho$  and input feedforward passivity(IFP) index  $\nu$  can be read as in Table I. Clearly, the boundary of passivity indices is moving due to the data is selected over different time intervals. With sufficient large data set, the passivity indices are expected to converge to certain fixed points. To maximize the passivity indices, the numerical optimization method is introduced in below.

## 5 Numerical Methods for Experimental Passivity Optimization

### 5.1 Hooke and Jeeve's method

If the system models are well established and accurate, optimization problems can be solved efficiently using the evaluation of the derivatives of the cost functions [9, 10, 11]. However, if the system dynamics and the cost function is greatly affected by nonlinearity or cannot be represented explicitly, the derivative-based optimization methods could terminate far from the optimum or even fail. In this case, one can choose direct methods involving only function evaluations as good alternatives, such as Rosenbrock [12], Hooke and Jeeves [13], and Nelder and Mead [14, 15].

Hooke and Jeeve's method [16, 17] is a numerical optimization method that does not require the gradient of the cost function or system performances. It is useful when the optimization problem is based on functions that are not continuous or not differentiable. Hooke and Jeeve's method works by creating a set of search directions iteratively. The created search directions span the search space such that starting from any point in the search space, any other point in the search space can be reached by traversing along these search directions only. One benefit is that only the current point and the next exploratory move is needed during the search, therefore the algorithm is efficient in terms of memory space used.

In the Hooke and Jeeve's method, a combination of exploratory moves and heuristic pattern moves is made iteratively. To maximize a cost function  $f(x)$  with  $x \in R^n$ , the procedure of Hooke and Jeeve's method is to first explore the neighborhood of current base point  $x_k$  in every direction with a small increment  $\epsilon_i$ . With variables in all directions considered, a new base point  $x_{k+1}$  will be reached to maximize  $f(x)$ . If no function maximization is achieved, the step length  $\epsilon_i$  will be reduced, otherwise a pattern move attempting to speed up the search will be made.

Initial Point	[1, 1]	[2, -1]	[-1, 1]	[-1, -1]
Optimal Point	[3, 2]	[3.58, -1.85]	[-2.81, 3.13]	[-3.78, -3.28]

Table 2: Hooke and Jeeve’s method optimization

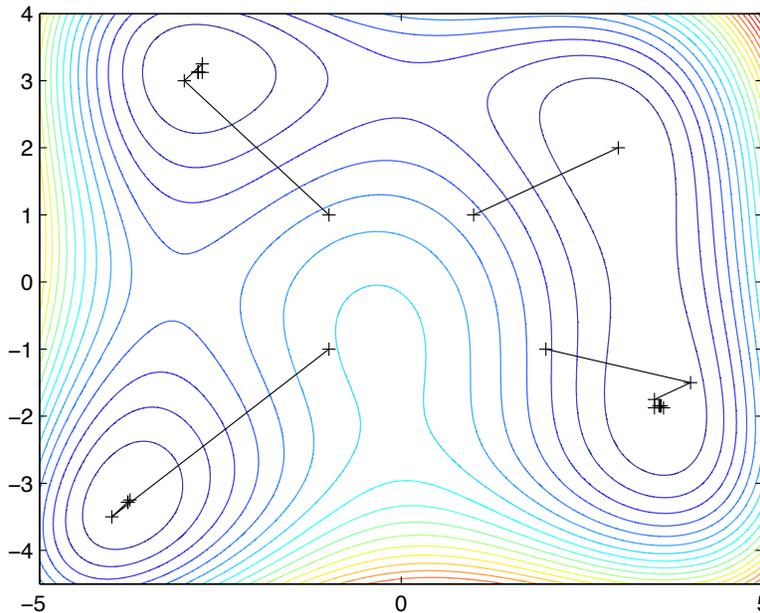


Figure 4: Hooke and Jeeve’s method optimization

The following example illustrates how Hooke and Jeeve’s methods works. Consider the Himmelblau function [17]

$$\min f(x_1, x_2) = (x_1^2 + x_2 - 11)^2 + (x_1 + x_2^2 - 7)^2$$

We choose four different initial points  $[1, 1], [2, -1], [-1, 1], [-1, -1]$ , then the optimization results given by the Hooke and Jeeve’s method will converge to four different local minimum, see Table 2 and Fig. 4.

In our case of adaptive cruise control algorithm, when the lower level controller applies throttle control or braking control, a proper engine speed is needed to generate the desired engine torque. This is performed by interpolating the data from an experimentally determined lookup table, called inverse engine map. To optimize the performance of such ACC algorithm, Hooke and Jeeve’s method can be used as a derivative-free or black-box method.

## 5.2 Optimization of Passivity Indices

We focus on the PI controller in the throttle control unit, see Fig. 5. The parameters set  $k_p, k_i$  in the PI controller consist of the search space in Hooke and Jeeve’s method. Then the optimization

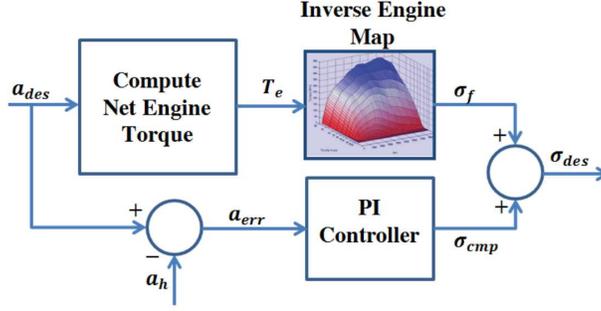


Figure 5: Throttle Controller

problem can be stated as follows,

$$\max_{x=[k_p, k_i]^T} \rho(u, y(u, x)) \quad (14)$$

The simulation results are shown below. The velocity of the lead vehicle is designed to be a sinusoid wave, however the velocity of host vehicle is not sinusoid according to the simulations.

The initial search point  $(K_p, K_i) = (1.5, 40)$ . After about 30 iterations, Hooke and Jeeve's method stops at the optimal point  $(K_p, K_i) = (3.75, 10)$  with maximal  $\rho = 0.98914$

Initial, intermediate and end points and the host vehicle behavior are shown in Fig. 6. Notice that at the intermediate point there might be jitters along the velocity trajectory when the host vehicle is try to catch up with the lead vehicle. Comparing to smooth trajectories, jitter curves does not fit the lead vehicle velocity curve very well. In terms of passivity, it means the passivity indices are not optimal.

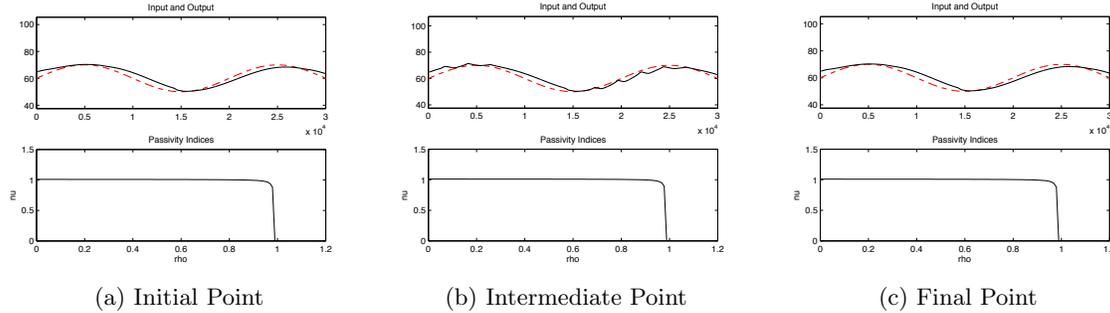


Figure 6: Passivity Indices Optimization

Due to physical constraints, the speed of the lead vehicle can not change dramatically. In another word, if we decompose the input signal(speed) into Fourier series, the coefficients of high frequency parts can be neglected. In the simulations, we select three different frequencies and make the speed of the lead vehicle sinusoid waves with corresponding frequencies, see Fig. 7. The optimization results are different in each case (Table 3), but overall they can give a loose bound of suboptimality and a good intuition for our problem.

	$f = 0.01$	$f = 0.05$	$f = 0.1$
Non-optimized Gains	$K_p = 1.5, K_i = 40$	$K_p = 1.5, K_i = 40$	$K_p = 1.5, K_i = 40$
Optimized Gains	$K_p = 2.625, K_i = 10$	$K_p = 1.5, K_i = 0$	$K_p = 0.375, K_i = 40$

Table 3: Optimization of  $K_p, K_i$

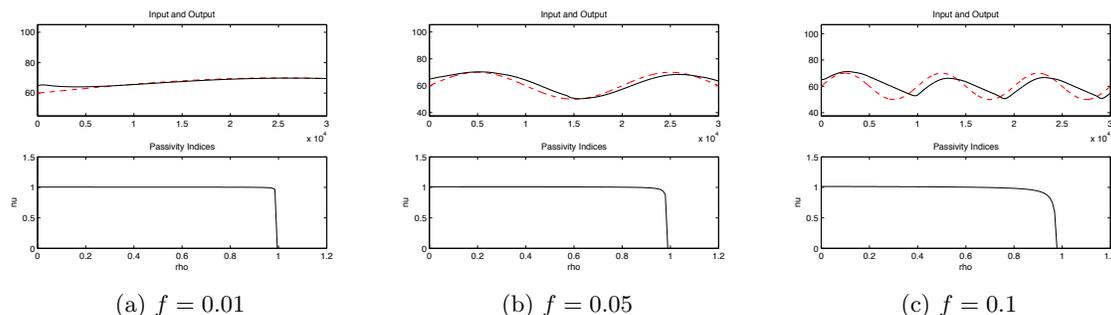


Figure 7: Simulations with sinusoidal inputs of different frequencies

## 6 Conclusions

In this paper, methods of determining passivity indices are considered. Besides the traditional analytical methods, the experimental passivity testing method is also discussed. Particularly, we consider testing passivity indices for vehicle systems with adaptive cruise control (ACC) algorithm, and maximizing passivity indices through a numerical optimization method, the Hooke and Jeeves method. Simulations on a virtual car platform are given to demonstrate the results.

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